Equilibrium of disordered systems: constructing the appropriate valleys in each sample via strong-disorder renormalization in configuration space

Equilibrium of disordered systems: constructing the appropriate valleys in each sample via strong-disorder renormalization in configuration space

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Abstract
To describe the equilibrium properties of disordered systems and the possible
emergence of various 'phases' at low temperature, we adopt here the 'broken-
ergodicity' point of view advocated in particular by Palmer (1982 *Adv. Phys.*
31 669): the aim is then to construct the valleys of configurations that
become separated by diverging barriers and to study their relative weights,
as well as their internal properties. To characterize the slow non-equilibrium
dynamics of disordered systems, we have recently introduced (Monthus and
Garel 2008 *J. Phys.* A **41** 255002, *Preprint* arXiv:0804.1847)) a strong-disorder
renormalization (RG) procedure in configuration space, based on the iterative
elimination of the smallest barrier remaining in the system. In the present
paper, we show how this renormalization procedure allows us to construct
the longest-lived valleys in each disordered sample, and to obtain their free-
energies, energies and entropies. This explicit RG formulation is very general
since it can be defined for any master equation, and it gives new insights into
the main ingredients of the droplet scaling picture. As an application, we have
followed numerically the RG flow for the case of a directed polymer in a two-
dimensional random medium to obtain histograms of the free-energy, entropy
and energy differences between the two longest-lived valleys in each sample.

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1. Introduction

In statistical physics, any large-scale universal behaviour is expected to come from some
underlying renormalization (RG) procedure that eliminates all the details of microscopic
models. In the presence of quenched disorder, interesting universal scaling behaviours
occur both at phase transitions (as in pure systems) but also in the low-temperature
disorder-dominated phases. Since the main property of frozen disorder is to break the
translational invariance, the most natural renormalization procedures that allow us to

describe spatial heterogeneities are *a priori* real-space RG procedures. Among real-space renormalization procedures that have been introduced for pure systems [1], Migdal–Kadanoff block renormalizations [2] play a special role, because they can be considered either as approximate renormalization procedures on hypercubic lattices, or as exact renormalization procedures on certain hierarchical lattices [3, 4]. They have thus been used to study various disordered models, such as the diluted Ising model [5], ferromagnetic random Potts model [6–8] and spin-glasses [9–15]. For the case of spin-glasses, the main output of these Migdal–Kadanoff RG studies is that the probability distribution $P_L(J)$ of the effective coupling J on scale L satisfies a closed RG equation whose solution flows towards a fixed shape with a scale-dependent width L^θ that defines the droplet exponent θ. To adapt this idea to hypercubic lattices, the notion of 'domain-wall RG' has been developed [16–19], where the droplet exponent θ is defined via the change of the free-energy of a given sample when periodic boundary conditions are replaced by antiperiodic boundary conditions. This definition can be used numerically in dimensions $d = 2, 3$ to obtain a fixed shape for the probability distribution of this free-energy difference when rescaled by L^θ [16–18]. Although this result is expected to come from some underlying RG procedure, we are not aware of an explicit principle to define a real-space RG procedure for this distribution for hypercubic lattices (apart from the Migdal–Kadanoff approximation that actually replaces the hypercubic lattice by a hierarchical lattice). Because of the difficulties to formulate an appropriate coarse-graining in real space (see the discussion in section VI of [19]), Fisher and Huse [19] have preferred to formulate their droplet scaling theory as 'a scaling ansatz for the nature, statistics, energetics and dynamics of the low-lying large scale excitations' [...] 'using RG ideas only to justify our scaling assumptions'.

The aim of the present paper is to show that the universal scaling behaviours that appear in the low-temperature disorder-dominated phases of disordered systems can be understood within an explicit and consistent RG procedure *in configuration space*. We adopt here the 'broken-ergodicity' point of view [20, 21] (see more details in section 2.1), so that the aim of the RG procedure is to construct the valleys of configurations that are separated by large barriers. To study the non-equilibrium dynamics of disordered systems, we have recently introduced a strong-disorder RG procedure to characterize the statistics of barriers on various scales and the extremely-slow logarithmic growth of the coherence length $l(t) \sim (\ln t)^{1/\psi}$ [22, 23]. In the present paper, we show how this method can be used to construct the valleys that are separated by barriers above a given threshold and to study their statistical properties.

This paper is organized as follows. In section 2, we briefly recall the 'broken-ergodicity' point of view and explain how a coarse-grained dynamics between valleys allows us to obtain their free-energy differences. In section 3, we describe the explicit strong-disorder RG rules for random master equations and discuss the properties of the valleys that emerge. In particular, we explain how the free-energy differences between valleys are constrained by a conservation law for ratios of renormalized transition rates along the RG flow. In section 4, we describe the 'simplified RG rules' that are valid at large scales near 'infinite-disorder fixed points', and that correspond to the notion of quasi-equilibrium inside each valley. Our conclusions are summarized in section 5.

2. Defining valleys via some coarse-grained dynamics

2.1. Reminder on the 'ergodicity/broken-ergodicity' point of view in statistical physics [20]

The statistical physics of equilibrium is based on Boltzmann's ergodic principle, which states the equivalence between the 'time average' of any observable A over a sufficiently long time

t by an 'ensemble average' over microscopic configurations c of energies $e(c)$

$$\frac{1}{t} \int_0^t d\tau \, A(\tau) \underset{t \to \infty}{\simeq} \sum_c A(c) p_{eq}(c) \tag{1}$$

where $p_{eq}(c)$ represents the Boltzmann measure at temperature $T = 1/\beta$

$$p_{eq}(c) = \frac{e^{-\beta e(c)}}{Z_{\text{tot}}} \tag{2}$$

and where Z_{tot} is the partition function

$$Z_{\text{tot}} \equiv \sum_c e^{-\beta e(c)}. \tag{3}$$

Even if historically and physically this dynamical interpretation of the equilibrium is crucial, it is sometimes a bit 'forgotten'. In particular, to discuss the appearance of low-temperature symmetry broken phases in pure systems such as ferromagnets, it has become usual to reason only *statically* in terms of the properties of the Boltzmann measure in the thermodynamic limit, but this way of thinking usually involves some 'cheating', in the sense that one adds an infinitesimal magnetic external field, or some special boundary conditions for the spins to select the possible 'pure states' that are obvious from the very beginning. For disordered systems, many discussions of the equilibrium are based on the same purely 'static' way of thinking, but they face the very essential problem that whenever disorder induces some frustration, the possible 'pure states' are not at all obvious because they depend on the realization of the randomness. In particular for spin-glasses (see the reviews [24]) which constitute the most studied frustrated disordered system, there exists a long-standing controversy on the nature and the number of pure states, as well as on the definition of appropriate order parameters. The main descriptions are the mean-field-inspired 'replica-symmetry-breaking' point of view [25], the renormalization-inspired 'droplet' scaling picture [17–19], the more recent numerically-inspired 'trivial-non trivial' [26] and 'state hierarchy of correlated spin domains' [27, 28] descriptions. As a consequence, a purely static definition of pure states based on the infinite-volume limit of finite-volume Boltzmann measures, which is simple for ferromagnets when the answer is known in advance, turns out to become very subtle for frustrated disordered systems where the number and the nature of pure states are not known in advance (see, for instance, the discussions in [29, 30]). We thus feel that for such systems a much clearer physical description can be achieved by returning to the 'historical' point of view of statistical mechanics where the equilibrium is considered as the stationary measure of some *dynamics*, so that the question on the number of 'phases' for the equilibrium becomes a question of ergodicity breaking for the dynamics: are there valleys in configuration space that become separated by diverging barriers in the thermodynamic limit and that keep nevertheless finite free-energy differences? This broken-ergodicity point of view is actually also the 'historical' point of view for the spin-glass problem. In their original paper [31], Edwards and Anderson have defined their order parameter by the following sentence: ' if on one observation a particular spin is $S_i(0)$, then if it is studied again a long time later, there is a non-vanishing probability that $S_i(t)$ will point in the same direction'. The importance of this definition has been further emphasized by Anderson in [32]: 'if the spins are going to polarize in a particular random function [. . .], we had better not try to characterize the order by some kind of long-ranged order in space, or by some kind of order parameter defined in space, but *we must approach it from a pure non-ergodic point of view, as a long-range order in the time alone*: if the system has a certain order at $t = 0$, then as $t \to +\infty$ there remains a finite memory of that order':

$$q_i = \lim_{t \to \infty} \langle S_i(0) S_i(t) \rangle. \tag{4}$$

Many works have then tried to characterize the phase-space structure of spin-glasses via dynamical studies, in particular by measuring numerically the distance between two configurations that are submitted to the same thermal noise to detect the presence of different valleys that become separated by diverging barriers in the thermodynamic limit [33]. Since the works on spin-glasses that are based on a 'dynamical' point of view of the equilibrium are too numerous to be summarized here, we will stop here this reminder, and we refer the reader to the papers of Palmer where the issues related to 'broken-ergodicity' are discussed in detail, both for statistical physics models in general [20] and for spin-glasses in particular [20, 21]. Once one has adopted this 'broken-ergodicity' point of view, the aim is to construct the valleys that tend to confine the dynamics in the thermodynamic limit. The starting point is thus some microscopic dynamics in phase space.

2.2. Microscopic dynamics with detailed balance

In statistical physics, it is convenient to define the dynamics via a master equation describing the evolution of the probability $p_t(c)$ to be in a microscopic configuration c at time t,

$$\frac{\mathrm{d}p_t(c)}{\mathrm{d}t} = \sum_{c'} p_t(c')w(c' \to c) - p_t(c)w_{\text{out}}(c). \tag{5}$$

The notation $w(c' \to c)$ represents the transition rate per unit time from configuration c' to c, and the notation

$$w_{\text{out}}(c) \equiv \sum_{c'} w(c \to c') \tag{6}$$

represents the total exit rate out of configuration c. To ensure that any finite system will converge towards thermal equilibrium in the limit of infinite time, we consider as usual that the transition rates satisfy the detailed balance property

$$\mathrm{e}^{-\beta e(c)}w(c \to c') = \mathrm{e}^{-\beta e(c')}w(c' \to c). \tag{7}$$

Then, the Boltzmann equilibrium distribution of equation (2) is the stationary solution of the master equation of equation (5).

2.3. Notion of coarse-grained dynamics between valleys

We now assume that the microscopic master equation of equation (5) can be coarse-grained into a renormalized master equation

$$\frac{\mathrm{d}P_t(C)}{\mathrm{d}t} = \sum_{C'} P_t(C')W(C' \to C) - P_t(C)W_{\text{out}}(C) \tag{8}$$

between 'renormalized configurations' denoted by C (not to be confused with the microscopic configurations denoted by c) in terms of renormalized transition rates W (not to be confused with the microscopic transition rates denoted by w). An explicit procedure to renormalize the master equation will be discussed in detail in section 3, but here, in the remaining of this section, our aim is to discuss the general meaning of any such coarse-grained dynamics. From now on, 'renormalized configurations' will be called 'valleys' to simplify the formulation.

2.4. Definition of the free-energies $F(C)$ of valleys

The stationary solution $P_{st}(C)$ of the renormalized master equation of equation (8) is then fixed (up to a normalization constant) by the ratios of the renormalized transition rates

$$\frac{P_{st}(C)}{P_{st}(C')} = \frac{W(C' \to C)}{W(C \to C')}. \tag{9}$$

In the statistical physics of equilibrium, it has been understood since Einstein that the probability of some fluctuation is determined by the free-energy cost $\Delta F(\text{Fluct})$ of this fluctuation

$$\text{Prob}_{eq}(\text{Fluct}) \propto e^{-\beta \Delta F(\text{Fluct})}. \tag{10}$$

This property is nowadays considered as sufficiently 'fundamental' to be used to extend the notion of free-energy to non-equilibrium situations: the 'out-of-equilibrium' free-energy is then defined from the large deviation function that govern the probability of fluctuations (see the recent review [34] and references therein for more detailed explanations and examples).

For our present problem in any finite system, the coarse-grained master equation will converge towards the stationary distribution $P_{st}(C)$ in the limit of infinite time. It is thus natural to define the free-energy $F(C)$ of the valley C from the stationary solution $P_{st}(C)$ of the renormalized master equation by

$$P_{st}(C) = \frac{e^{-\beta F(C)}}{Z_R} \tag{11}$$

with the normalization Z_R ensuring that $\sum_C P_{st}(C) = 1$. In particular, the ratios of the renormalized transition rates of equation (9) determines directly the free-energy differences between valleys

$$e^{-\beta(F(C)-F(C'))} = \frac{W(C' \to C)}{W(C \to C')}. \tag{12}$$

It is very important to stress here that the free-energies of the valleys have been defined in equations (11) and (12) from the coarse-grained dynamics between valleys, i.e. from the *inter-valleys* dynamics. But we have made no statement yet about what happens inside each valley, because the intra-valley properties actually depend on the explicit principle that is used to renormalize the master equation. We will thus rediscuss this point in the following sections for the explicit strong-disorder renormalization procedure that we will use to renormalize the master equation.

2.5. Inter-valleys entropy S^{inter} and valleys weights statistics

Once the stationary distribution $P_{st}(C)$ of the renormalized master equation is known, it is natural to introduce the following inter-valleys entropy S^{inter},

$$S^{\text{inter}} = -\sum_C P_{st}(C) \ln P_{st}(C). \tag{13}$$

This quantity is called 'complexity' in [20, 21], but since this word is used with different meanings, we will keep the name 'inter-valley entropy' for clarity.

To better characterize the statistics of the valleys weights, it is interesting to introduce the generalized moments [35]

$$Y_k = \sum_C [P_{st}(C)]^k \tag{14}$$

(one has the normalization $Y_{k=1} = 1$, and the entropy S^{inter} of equation (13) corresponds to $S^{\text{inter}} = -\partial Y_k|_{k=1}$). Results on the valleys weights statistics in mean field spin-glasses (in the random energy model and in the SK model) and in other statistical physics models are reviewed in [35]. More detailed studies on probability distributions of Y_k can be found in [36].

3. Strong-disorder renormalization of master equations

In the present section, we explain why the strong-disordered renormalization of master equation introduced in [22] is the appropriate way to construct the coarse-grained dynamics between valleys starting from the microscopic master equation. Strong-disorder renormalization (see [37] for a review) is a very specific type of RG that has been first developed in the field of quantum spins: the RG rules of Ma and Dasgupta [38] have been put on a firm ground by DS Fisher who introduced the crucial idea of 'infinite-disorder' fixed point where the method becomes asymptotically exact, and who computed explicitly exact critical exponents and scaling functions for one-dimensional disordered quantum spin chains [39]. This method has thus generated a lot of activity for various disordered quantum models [37]. It has then been successfully applied to various classical disordered dynamical models, such as random walks in random media [40, 41], reaction–diffusion in a random medium [42], coarsening dynamics of classical spin chains [43], trap models [44], random vibrational networks [45], absorbing state phase transitions [46], zero range processes [47] and exclusion processes [48]. In all these cases, the strong-disorder RG rules have been formulated *in real space*, with specific rules depending on the problem. For more complex systems where the formulation of strong-disorder RG rules has not been possible in real space, we have recently proposed in [22] a strong-disorder RG procedure *in configuration space* that can be defined for any master equation. In the remaining of this section, we first recall this procedure, and then discuss its consequence for the free-energy differences that can be extracted from the renormalized transition rates (see equation (12)), and for the energies of the valleys.

3.1. Reminder on the 'full' strong-disorder RG rules [22]

For dynamical models, the aim of any renormalization procedure is to integrate over 'fast' processes to obtain effective properties of 'slow' processes. The general idea of 'strong renormalization' for dynamical models consists in eliminating iteratively the 'fastest' process. The RG procedure introduced in [22] can be summarized as follows (see [22, 23] for more explanations):

(1) Find the configuration C^* with the largest exit rate W_{out}^*

$$W_{\text{out}}^* = W_{\text{out}}(C^*) \equiv \max_C [W_{\text{out}}(C)]. \tag{15}$$

(2) Find the neighbours $(C_1, C_2, \ldots, C_n)$ of configuration C^*, i.e. the configurations that were related via positive rates $W(C^* \to C_i) > 0$ and $W(C_i \to C^*) > 0$ to the decimated configuration C^* (here we will assume for the simplicity of the discussion, and because it is usually the case in statistical physics models, that if a transition has a strictly positive rate, the reverse transition also has a strictly positive rate). For each neighbour configuration C_i with $i \in (1, \ldots, n)$, update the transition rate to go to the configuration C_j with $j \in (1, \ldots, n)$ and $j \neq i$ according to

$$W^{\text{new}}(C_i \to C_j) = W(C_i \to C_j) + W(C_i \to C^*) \times \pi_{C^*}(C_j), \tag{16}$$

where the first term represents the 'old' transition rate (possibly zero), and the second term represents the transition via the decimated configuration C^*: the factor $W(C_i \to C^*)$ takes into account the transition rate to C^* and the term

$$\pi_{C^*}(C_j) = \frac{W(C^* \to C_j)}{W_{\text{out}}(C^*)} \tag{17}$$

represents the probability to make a transition towards C_j when in C^*. The $2n$ rates $W(C^* \to C_i)$ and $W(C_i \to C^*)$ then disappear with the decimated configuration C^*.

Note that the rule of equation (16) has been recently proposed in [49] to eliminate 'fast states' from various dynamical problems with two very separated time scales. The physical interpretation of this rule is as follows: the time spent in the decimated configuration C^* is neglected with respect to the other time scales remaining in the system. The validity of this approximation within the present framework will be discussed in detail below in 3.2. The interesting equivalence of the rule of equation (17) with the well-known 'adiabatic' approximation [49] will be recalled in section 3.4 to clarify what is really assumed physically for the decimated renormalized configuration C^*.

(3) Update the exit rates out of the neighbours C_i, with $i = 1, \ldots, n$ either with the definition

$$W_{\text{out}}^{\text{new}}(C_i) = \sum_C W^{\text{new}}(C_i \to C) \tag{18}$$

or with the equivalent rule that can be deduced from equation (16)

$$W_{\text{out}}^{\text{new}}(C_i) = W_{\text{out}}(C_i) - W(C_i \to C^*) \frac{W(C^* \to C_i)}{W_{\text{out}}^*}. \tag{19}$$

The physical meaning of this rule is as follows. The exit rate out of the configuration C_i decays because the previous transition towards C^* can lead to an immediate return towards C_i with probability $\pi_{C^*}(C_i) = \frac{W(C^* \to C_i)}{W_{\text{out}}^*}$. After the decimation of the configuration C^*, this process is not considered as an 'exit' process anymore, but as a residence process in the configuration C_i. This point is very important to understand the meaning of the renormalization procedure: the remaining configurations at a given renormalization scale are 'formally' microscopic configurations of the initial master equation (equation (5)), but each of these remaining microscopic configuration actually represents some 'valley' in configuration space that takes into account all the previously decimated configurations.

(4) Return to point (1).

Note that, in practice, the renormalized rates $W(C \to C')$ can rapidly become very small as a consequence of the multiplicative structure of the renormalization rule of equation (16). This means that the appropriate variables are the logarithms of the transition rates, that we will call 'barriers' in the remainder of this paper. The barrier $B(C \to C')$ from C to C' is defined by

$$B(C \to C') \equiv -\ln W(C \to C') \tag{20}$$

and similarly the exit barrier out of configuration C is defined by

$$B_{\text{out}}(C) \equiv -\ln W_{\text{out}}(C). \tag{21}$$

Note that a very important advantage of this formulation in terms of the renormalized transition rates of the master equation is that the renormalized barriers take into account the true 'barriers' of the dynamics, whatever their origin which can be either energetic or entropic.

3.2. Notion of 'infinite-disorder fixed point' and asymptotic exactness of the RG rules

As mentioned above, the approximation made in the renormalization rule of equation (16) consists in neglecting the time spent in the decimated configuration C^* with respect to the other time scales remaining in the system. In the present framework, this means that the maximal exit rate chosen in equation (15) should be well separated from the exit rates of the neighbouring configurations C_i. The crucial idea of 'infinite-disorder fixed point' [37, 39] is that even if this approximation is not perfect during the first steps of the renormalization, this approximation will become better and better at large time scale if the probability distribution of the remaining

exit rates becomes broader and broader upon iteration. More precisely, if the renormalization scale Γ is defined as the exit barrier of the last decimated configuration C^*

$$\Gamma = B_{\text{out}}(C^*) \equiv -\ln W^*_{\text{out}} \tag{22}$$

one expects that the probability distribution of the remaining exit barrier $B_{\text{out}} \geqslant \Gamma$ will converge towards some scaling form

$$P_\Gamma(B_{\text{out}} - \Gamma) \underset{\Gamma \to \infty}{\simeq} \frac{1}{\sigma(\Gamma)} \hat{P}\left(\frac{B_{\text{out}} - \Gamma}{\sigma(\Gamma)}\right) \tag{23}$$

where $\hat{P}$ is the fixed point probability distribution, and where $\sigma(\Gamma)$ is the appropriate scaling factor for the width. The notion of 'infinite-disorder fixed point' means that the width $\sigma(\Gamma)$ grows indefinitely with the renormalization scale Γ,

$$\sigma(\Gamma) \underset{\Gamma \to \infty}{\simeq} +\infty. \tag{24}$$

Whenever this 'infinite-disorder fixed point' condition is satisfied, the strong-disorder renormalization procedure becomes asymptotically exact at large scales. In previously known cases of infinite-disorder fixed points where calculations can be done explicitly [37], the scale $\sigma(\Gamma)$ has been found to grow linearly $\sigma(\Gamma) \simeq_{\Gamma \to \infty} \Gamma$. This behaviour means that the cut-off Γ is the only characteristic scale and thus describes some critical point [37]. For the present procedure concerning the dynamics in disordered models, this property means some 'criticality in the time direction', i.e. the absence of any characteristic time scale between the microscopic scale and the macroscopic equilibrium time of the full-disordered sample (see [23] for more detailed discussions).

For the present strong-disorder renormalization of a master equation, the convergence towards an 'infinite-disorder fixed point' will depend on the initial condition of the transition rates, i.e. on the model (and on the temperature if there are phase transitions). However, the form of the RG rules of equation (16) is sufficiently similar to the usual Ma–Dasgupta rules [37] to think that the convergence towards some infinite-disorder fixed point should be realized in a very broad class of disordered systems in their glassy phase. In practice, it should be checked numerically for each model of interest. We refer to [22, 23] where this scenario has been checked numerically for the dynamics of a directed polymer in a two-dimensional random medium.

3.3. RG flow for the free-energy differences between valleys

We have explained above how the free-energy difference between two valleys can be obtained from the ratio of the two transition rates between them (equation (12)). It is thus convenient to introduce the following notation for these ratios,

$$R(C_i \to C_j) \equiv \frac{W(C_i \to C_j)}{W(C_j \to C_i)}, \tag{25}$$

to see how they evolve upon renormalization. In the initial condition (RG scale $\Gamma = 0$) corresponding to the microscopic master equation of equation (5), the values of all these ratios are fixed by the detailed balance condition of equation (7)

$$R^{\Gamma=0}(c_i \to c_j) \equiv \frac{w(c_i \to c_j)}{w(c_j \to c_i)} = e^{-\beta(e(c_j)-e(c_i))} \tag{26}$$

in terms of the energies $e(c_i)$ and $c(e_j)$ of the microscopic configurations (c_i, c_j).

From the full RG rules of equations (16) and (17), we obtain the following RG rule for the ratio:

$$R^{\text{new}}(C_i \to C_j) = \frac{W^{\text{new}}(C_i \to C_j)}{W^{\text{new}}(C_j \to C_i)} = \frac{W(C_i \to C_j) + W(C_i \to C^*) \times \frac{W(C^* \to C_j)}{W_{\text{out}}(C^*)}}{W(C_j \to C_i) + W(C_j \to C^*) \times \frac{W(C^* \to C_i)}{W_{\text{out}}(C^*)}} \tag{27}$$

$$= \frac{[W_{\text{out}}(C^*)W(C_j \to C_i)]R(C_i \to C_j) + [W(C_j \to C^*)W(C^* \to C_i)]R(C_i \to C^*)R(C^* \to C_j)}{[W_{\text{out}}(C^*)W(C_j \to C_i)] + [W(C_j \to C^*)W(C^* \to C_i)]}. \tag{28}$$

Let us first consider the first RG step: since the initial value of equation (26) has actually a factorized form, one obtains that the product of the second term in the numerator is simply $R(C_i \to C^*)R(C^* \to C_j) = e^{-\beta(e(c_j)-e(c_i))} = R(C_i \to C_j)$. As a consequence in equation (28), the factor $R(C_i \to C_j)$ can be factorized and what remains in the numerator is exactly equal to the denominator. This simplification is actually valid at all steps of the RG procedure by recurrence. We thus obtained that the ratios $R(C_i \to C_j)$ are actually 'conserved' by the RG flow in the following sense

$$R^{\text{new}}(C_i \to C_j) = R^{\Gamma=0}(C_i \to C_j) = e^{-\beta(e(c_j)-e(c_i))} \tag{29}$$

where c_i and c_j are the microscopic configurations that label the renormalized valleys C_i and C_j respectively.

Since the free-energy difference between two valleys can be obtained from the ratio of the two transition rates between them (equation (12)), we finally obtain

$$e^{-\beta(F(C_j)-F(C_i))} = R^{\text{new}}(C_i \to C_j) = e^{-\beta(e(c_j)-e(c_i))}, \tag{30}$$

i.e. the free-energy difference between the two renormalized valleys C_i and C_j is simply given by the difference of energy of the two microscopic configurations c_i and c_j that label the renormalized valleys. This property indicates that the statistics of free-energy differences between renormalized valleys is the same as the statistics of energy differences between 'good' microscopic configurations. This property is usually assumed within the droplet scaling picture [18, 19] where the low-temperature phase is expected to be governed by a zero-temperature fixed point. Within the strong-disorder RG procedure, this property emerges as a consequence of a conservation law for the ratio $R(C_i \to C_j)$ of the transition rates between two renormalized configurations.

As a final remark, we should stress here that this conservation law does not mean that the free-energy differences remain the same as the temperature T varies, because the valleys C_i and C_j that survive during the RG procedure are temperature dependent, i.e., the microscopic configurations c_i and c_j that label the longest-lived renormalized valleys will change with the temperature. Within our present RG framework, the 'chaos properties' with respect to temperature changes, which are expected within the droplet scaling theory [18, 19], should correspond to the sensitivity of the RG flow with respect to changes in the initial condition, as in Migdal–Kadanoff real-space RG procedures (see [10, 13–15] and references therein). However a detailed study of these 'chaos' properties for the present strong-disorder procedure goes beyond the present work and these effects will not be discussed further here.

3.4. Interpretation of the RG rules as an 'adiabatic approximation'

As explained in [49] where the rule of equation (16) has been proposed to eliminate 'fast states' in dynamical problems with two very separated time scales, the prescription to obtain the new transition rates actually corresponds to an 'adiabatic approximation'. Let us explain this point within our present notations since it has important consequences for the present coarse-graining procedure.

The idea of the 'adiabatic approximation' [49] is that 'fast' degrees of freedom adapt rapidly to ensure that the probability flow entering into them is exactly compensated by the probability flow emerging from them. Within our present notation where the 'fast' state is the configuration C^*, the probability flow entering into C^* from its neighbours $(C_1, \ldots, C_n)$ reads

$$J^{in}_{C^*}(t) = \sum_{i=1}^{n} P_t(C_i) W(C_i \to C^*) \tag{31}$$

whereas the probability flow emerging from C^* reads

$$J^{out}_{C^*}(t) = P_t(C^*) W_{out}(C^*) = P_t(C^*) \sum_{i=1}^{n} W(C^* \to C_i). \tag{32}$$

The adiabatic approximation consists in assuming that on time scales much larger than the typical relaxation time $1/W_{out}(C^*)$ of C^*, the occupation probability $P_t(C^*)$ adapts to ensure the global zero-flow condition $J^{in}_{C^*}(t) = J^{out}_{C^*}(t)$ leading to

$$P_t(C^*) = \frac{1}{W_{out}(C^*)} \sum_{i=1}^{n} P_t(C_i) W(C_i \to C^*), \tag{33}$$

i.e. the occupation probability $P_t(C^*)$ of the 'fast' configuration C^* now only varies slowly in time by following the external slow inputs $P_t(C)$ given by the 'slow' configurations C that are connected to it. Physically, this means that the 'fast' configuration C^* is equilibrated as much as it can in the 'out-of-equilibrium' environment produced by the 'slow' configurations.

The equivalence of this way of thinking with the renormalization rule of equation (16) is immediate by replacing the value of equation (33) into the evolution equation of a neighbour C_i of C^*:

$$\frac{dP(C_i)}{dt} = P_t(C^*) W(C^* \to C_i) + \sum_{C \neq C^*} P_t(C) W(C \to C_i) - P_t(C_i) W_{out}(C_i)$$

$$= \left[\frac{1}{W_{out}(C^*)} \sum_{j=1}^{n} P_t(C_j) W(C_j \to C^*) \right] W(C^* \to C_i)$$

$$+ \sum_{C \neq C^*} P_t(C) W(C \to C_i) - P_t(C_i) W_{out}(C_i)$$

$$= \sum_{C \neq C^*} P_t(C) \left[W(C \to C_i) + \frac{W(C \to C^*) W(C^* \to C_i)}{W_{out}(C^*)} \right]$$

$$- P_t(C_i) \left[W_{out}(C_i) - \frac{W(C_i \to C^*) W(C^* \to C_i)}{W_{out}(C^*)} \right]. \tag{34}$$

The terms between brackets [...] corresponds exactly to the renormalization rules of equations (16) and (19).

3.5. RG rule for the energies of valleys

As emphasized in [49], the 'adiabatic' interpretation shows that even if one eliminates the 'fast' modes to obtain the effective dynamics of 'slow' modes, the information on the 'fast' modes is actually not completely lost since its slow variation in time can be reconstructed via equation (33) from the dynamics of the 'slow' modes. Moreover, this point of view allows us to derive how the decimated valley C^* contributes to modify the energies of the neighbouring

renormalized valleys. Equation (33) can be interpreted as follows: after the elimination of C^*, when the system is in the renormalized configuration C_i, it is actually in the decimated configuration C^* with a temporal weight $\frac{W(C \to C^*)}{W_{\text{out}}(C^*)}$. As a consequence, the appropriate RG rule for the energies of the neighbouring valleys C_i upon the elimination of the configuration C^* reads

$$E^{\text{new}}(C_i) = \frac{E(C_i) + E(C^*)\frac{W(C_i \to C^*)}{W_{\text{out}}(C^*)}}{1 + \frac{W(C_i \to C^*)}{W_{\text{out}}(C^*)}}. \tag{35}$$

More generally, one may write similar RG equations for observables that are linear in the probabilities $P_t(C)$, but not for the entropy which involves logarithms. (The definition of intra-valleys entropies will be possible only with the simplified RG rules discussed in section 4.)

3.6. Discussion on the effects of 'bad decimations'

The interpretation in terms of an adiabatic approximation allows also to understand the effects of 'bad' decimations, defined as the cases where the exit rate out of C^* is not very well separated from the exit rates out of its neighbours. As explained in [49], there exists some commutativity of the RG rules in the following sense: the order of elimination of the 'fast' states is not important anymore when all the 'fast' states have been decimated. The reason is that in the adiabatic approximation, one may eliminates all the 'fast' states simultaneously by requiring that all the corresponding probability currents vanish simultaneously. The solution of this linear system is unique and the RG procedure simply corresponds to solve this system by substitution in a given order, but any other order would have given the same unique solution. This shows that when there are only two very different time scales as in [49], all 'fast' states can actually be eliminated at once. But in our present framework for disordered systems where a broad distribution of time scales exist, from the microscopic time scale of a single move to the equilibrium time of the whole system, it is not obvious at the beginning which set of microscopic configurations should be eliminated to obtain a consistent set of renormalized valleys whose exit rates is above some prescribed threshold. As a consequence, the iterative RG procedure that renormalizes the exit rates out of the surviving configurations is actually essential to determine which are the longest-lived renormalized valleys in each sample. Nevertheless, the 'commutativity' property is important, since it shows that 'bad decimations' will actually be corrected in the later stages of the renormalization when the neighbours will themselves be decimated. This phenomenon is very reminiscent of what has been found by DS Fisher for quantum spin chains in appendix E of [39] dedicated to the effects of a bad decimation: ' we recover exactly at a later stage from the errors made earlier'. This property explains why strong-disorder RG methods are usually much more accurate than one might think from the approximations involved in the first stages of the RG procedure where the probability distributions are not yet broad enough. The final picture is that at large RG scale Γ, where the probability distributions are broad enough to ensure a separation of scales between the decimated configuration and its neighbours: (i) the 'present errors' are small and of order $1/\Gamma$ (ii) the 'past errors' due to the first stages of the RG procedure have been cured by later decimations.

3.7. Example: directed polymer in a two-dimensional medium

As an example of application, we consider the directed polymer in a two-dimensional random medium (see [50] for a review), first introduced to model an interface in the low-temperature

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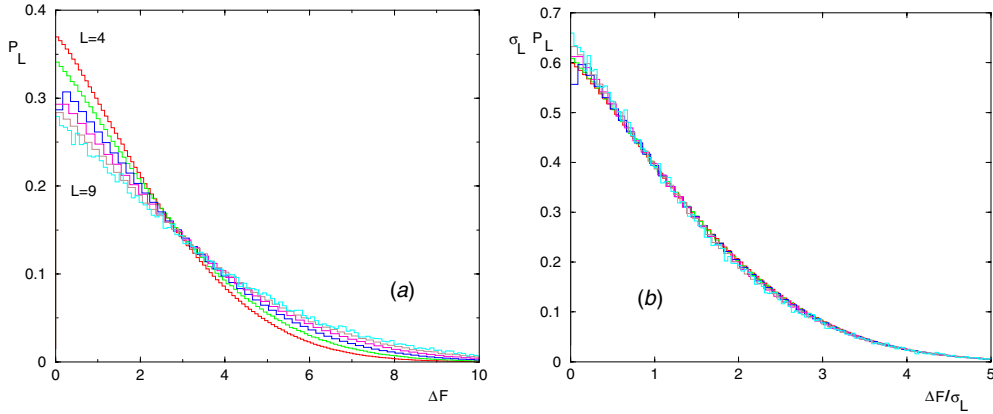
Figure 1. Statistics over the samples of the free-energy difference $\Delta F = F_2 - F_1$ between the two last valleys of a disordered sample. (*a*) Probability distribution $P_L(\Delta F)$ for the sizes $L = 4, 5, 6, 7, 8, 9$ (corresponding to $2^L \leqslant 512$ initial configurations) with a statistics of $n_s \geqslant 10^5$ disordered samples (we have data up to $L = 11$ with $n_s = 1200$ samples, but histograms are too noisy). (*b*) Same data after the rescaling by the width σ_L : $\sigma_L P_L$ as a function of $\Delta F/\sigma_L$ (see equation (36)).

phase of two-dimensional disordered ferromagnets [51]. The equilibrium is well described by the Fisher–Huse droplet theory [52] as checked by detailed numerical studies [52, 53] based on transfer matrix calculations of the partition function that allow us to study big sizes with a good statistics. Here we wish to follow numerically the strong-disorder RG procedure to obtain the properties of the longest-lived valleys. Since we work in configuration space, we will not be able to study big system sizes (see [23] for more details), but our aim is to see that the strong-disorder RG procedure can be applied consistently and yields appropriate results for the free-energy difference and for the weight statistics of the two longest-lived valleys. The numerical details are described in our previous works on the non-equilibrium dynamics [22, 23], where we have checked the convergence towards an infinite-disorder fixed point for the distribution of the renormalized barriers and where we have studied the distribution over samples of the equilibrium time. Here we present our results for the observables characterizing the valleys weights.

3.7.1. Histogram of the free-energy difference between the two last valleys. For each disordered sample of size L, we obtain via the strong-disorder renormalization the free-energy difference $\Delta F = F_2 - F_1$ between the two last valleys (see equation (12) and (30)). We show in figure 1(*a*) the probability distribution $P_L(\Delta F)$ over the samples of this free-energy difference for various sizes. As expected from the droplet scaling picture [18, 19], this probability distribution can be rescaled by its width σ_L

$$P_L(\Delta F) \sim \frac{1}{\sigma_L} \tilde{P}\left(u \equiv \frac{\Delta F}{\sigma_L}\right) \tag{36}$$

and the rescaled distribution $\tilde{P}$ is finite at the origin $\tilde{P}(0) > 0$: the rescaling of our data shown on figure 1(*b*) corresponds exactly to the expected shape for droplet free-energies distribution (see figure 2 of [19]). Of course, the sizes studied here $4 \leqslant L \leqslant 11$ are too small to obtain from the width $\sigma_L \sim L^\theta$ a precise measure of the droplet exponent which is exactly known to be $\theta = 1/3$ in the present case [51].
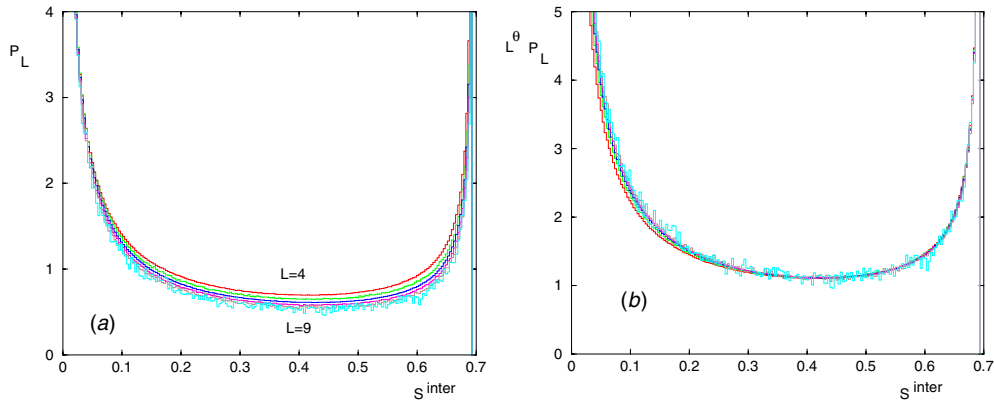
Figure 2. Statistics of the intervalley entropy $S^{\text{inter}} = -p \ln p - (1-p) \ln(1-p)$ for the two last valleys with $p = \frac{1}{1+e^{-\beta \Delta F}}$. (a) Probability distribution $P_L(S^{\text{inter}})$ for the sizes $L = 4, 5, 6, 7, 8, 9$. (b) Same data after the rescaling: $L^\theta P_L(S^{\text{inter}})$ with $\theta = 1/3$.
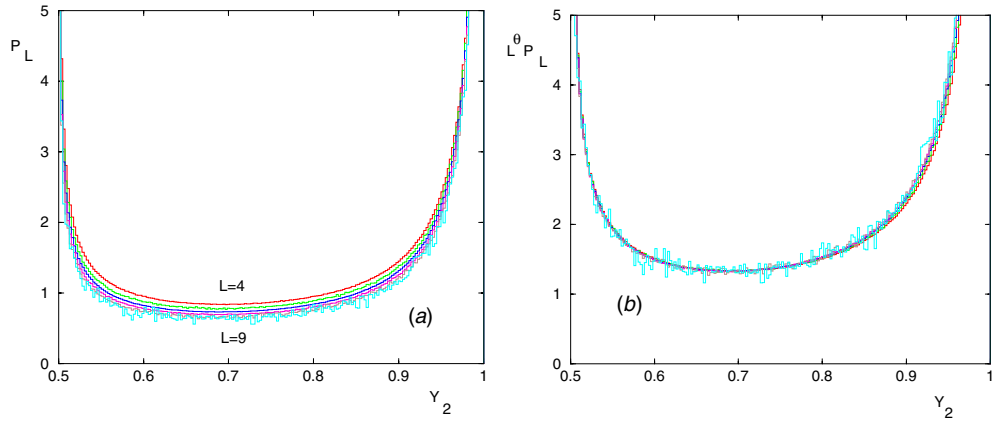
Figure 3. Statistics of $Y_2 = p^2 + (1-p)^2$ for the two last valleys with $p = \frac{1}{1+e^{-\beta \Delta F}}$. (a) Probability distribution $P_L(Y_2)$ for the sizes $L = 4, 5, 6, 7, 8, 9$. (b) Same data after the rescaling: $L^\theta P_L(Y_2)$ with $\theta = 1/3$.

3.7.2. Histograms of S^{inter} and Y_2 for the two last valleys.   The free-energy difference ΔF between the two last valleys determines their respective weights $(p, 1-p)$ in configuration space with

$$p = \frac{1}{1 + e^{-\beta \Delta F}}. \tag{37}$$

We show in figures 2 and 3 the histograms of the corresponding inter-valley entropy (equation (13))

$$S^{\text{inter}} = -p \ln p - (1-p) \ln(1-p) \tag{38}$$

and of the second generalized moment of equation (14)

$$Y_2 = p^2 + (1-p)^2 \tag{39}$$

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for various sizes. For a positive-droplet exponent $\theta > 0$, one expects that the corresponding distributions converge respectively towards $\delta(S^{\text{inter}})$ and $\delta(Y_2 - 1)$, with corrections of order $1/L^\theta$ on the intervals $0 < S^{\text{inter}} < \ln 2$ and $1/2 < Y_2 < 1$. We thus show in figures 2(b) and 3(b) the rescaled data by a factor $L^{1/3}$: we observe that the data collapse is better in the region of nearly degenerate weights between the two valleys, corresponding to S^{inter} near $\ln 2$ and to Y_2 near $1/2$. This seems to indicate that the corrections to the asymptotic scaling with the droplet exponent $\theta = 1/3$ are smaller in the region of small free-energy difference than in the region of large free-energy difference.

4. 'Simplified' RG rules near infinite-disorder fixed points

This section recalls the 'simplified RG rules' introduced in [23] and discuss their consequences for the valleys properties, in terms of quasi-equilibrium within each renormalized valleys.

4.1. Reminder on the simplified RG rules based the preferred exit channel [23]

Whenever the flow is towards some 'infinite-disorder' fixed point, where the distribution of renormalized exit barriers becomes broader and broader upon iteration (equations (23) and (24)), one expects that the exit rate out of the decimated configuration C^*

$$W_{\text{out}}(C^*) = \sum_{i=1}^{n} W(C^* \to C_i) \tag{40}$$

will actually be dominated by the preferred exit channel i_{pref} having the largest contribution in the sum of equation (40)

$$W_{\text{out}}(C^*) \simeq W(C^* \to C_{i_{\text{pref}}}), \tag{41}$$

i.e. one expects that the probability distribution $\pi_{C^*}(C_j)$ of equation (17) will become a delta distribution on the preferred exit channel up to exponentially small terms,

$$\pi_{C^*}(C_j) \simeq \delta_{j,i_{\text{pref}}} + \cdots. \tag{42}$$

The dominance of the preferred exit channel (equation (41)) near an infinite-disorder fixed point has been checked numerically in [23] for the case of the directed polymer in a two-dimensional random medium. However, we expect that it holds more generally for the following reasons. The RG rules with their characteristic multiplicative structure of equations (16) and (17) act directly on the transition rates $W(C_j \to C_i)$ between configurations, whereas the total exit rates W_{out} are derived quantities obtained by summing over the possible exit channels. The notion of convergence towards an infinite-disorder fixed point has been defined above by the property that the probability distribution of the remaining exit rates W_{out} becomes broader and broader. However, we expect that when it happens, it is because the probability distribution of the individual transition rates $W(C_j \to C_i)$ themselves becomes broader and broader (see [23] for more details).

The 'simplified RG procedure' introduced in [23] can be summarized as follows :

(1) the first point is the same as in the 'full RG rules' (equation (15)),

(2) among the neighbours $(C_1, C_2, \ldots, C_n)$ of configuration C^*, find the preferred exit channel i_{pref}. Update the transitions rates from the $(n - 1)$ non-preferred neighbours $i \neq i_{\text{pref}}$ towards i_{pref} by the approximated rule

$$W^{\text{new}}(C_i \to C_{i_{\text{pref}}}) \simeq W(C_i \to C_{i_{\text{pref}}}) + W(C_i \to C^*) \tag{43}$$

where the probability distribution $\pi_{C^*}(C_j)$ of the full RG rule of equations (16) and (17) has been replaced by the leading delta function of equation (42). Update the transitions rates from

i_{pref} towards the $(n-1)$ non-preferred neighbours $i \neq i_{\mathrm{pref}}$ by the full RG rule of equations (16) and (17)

$$W^{\mathrm{new}}(C_{i_{\mathrm{pref}}} \to C_i) = W(C_{i_{\mathrm{pref}}} \to C_i) + W(C_{i_{\mathrm{pref}}} \to C^*) \times \frac{W(C^* \to C_i)}{W_{\mathrm{out}}(C^*)}. \qquad (44)$$

Here the full rule is used because the ratio $\frac{W(C^* \to C_i)}{W_{\mathrm{out}}(C^*)}$ is small and should thus be evaluated correctly.

(3′) With the rule of equation (43), the exit rates out of the $(n-1)$ non-preferred neighbours $i \neq i_{\mathrm{pref}}$ do not have to be updated since the exit rate towards C^* has been completely transferred to i_{pref}. So the only update of exit rate is for the preferred neighbour i_{pref} via the definition of equation (18) or with the equivalent rule of equation (19).

(4) Return to (1).

As explained in [23], these simplified RG rules are interesting both from a computational point of view (they allow us to study bigger system sizes) and from a theoretical point of view to make the link with the idea of 'internal ergodicity' in each valley [20], as we now explain.

4.2. Interpretation in terms of quasi-equilibrium within valleys

In the studies on slowly relaxing systems such as disordered systems, glasses or granular media, it is usual to separate the dynamics into two parts: there are 'fast' degrees of freedom which rapidly reach local quasi-equilibrium plus a slow non-equilibrium part. Within the present strong-disorder renormalization in configuration space, these ideas can be applied directly as follows. To each time t, one may associate a set of valleys which are labelled by the surviving configurations at the RG scale $\Gamma = \ln t$. Since at large scale, the RG flow for the barrier distribution is towards some 'infinite disorder' fixed point, the different time scales are effectively very well separated. And thus asymptotically we recover the Palmer's decomposition into 'components' from which the escape probability is small and in which there is 'internal ergodicity', i.e. the interior of each valley has been able to equilibrate [20, 21]. Moreover, the slow non-equilibrium part of the dynamics corresponds to the evolution of the renormalized valleys with the RG scale: some valleys disappear and are absorbed by a neighbouring valley. Here again, we see that the RG procedure corresponds asymptotically to Palmer 'bifurcation cascade of components': see figure 4 of [20] where the 'bifurcation cascade' is drawn as a function of temperature for a fixed time. In our present framework, it is better to represent this 'bifurcation cascade' as a function of time for fixed temperature. This is because it has been understood since the papers of Palmer [20, 21] that disordered systems have some 'chaos' property with respect to temperature changes, i.e. the valleys that will emerge at large scales for different temperatures are not simply related (for more details on these chaos properties, see [13–15, 18, 19]).

To finish this discussion, we would like to emphasize a very important point: the asymptotic dominance of the preferred exit channel near the infinite-disorder fixed point is actually crucial to obtain quasi-equilibrium inside valleys. In particular, if the degeneracy between the second preferred exit channel and the first preferred exit channel could occur with a finite probability, then finite contributions of out-of-equilibrium situations at all scales would ruin the quasi-equilibrium approximation: the probability to be in a configuration C at time t would not depend only on its energy $E(C)$ and on the partition function of the renormalized valley it belongs to, but would be instead a very complicated function of all possible paths from C_0 to C with their appropriate dynamical weights. To better understand the importance of this discussion, it is useful to recall here a well-identified exception of the quasi-equilibrium idea, namely the symmetric Bouchaud's trap model in one dimension, where even in the

limit of arbitrary low temperature, the diffusion front in each sample consists in two delta peaks, which are completely out of equilibrium with each other [44]: the weights of these two delta peaks do not depend on their energies, but instead on the distances to the origin that determine the probability to reach one before the other (see [44] for more details). In this trap model, the reason is clear: whenever the particle escapes from a trap, it jumps either to the right or to the left with equal probabilities (1/2), i.e. the two possible exit channels are degenerate by the very definition of the model that imposes this symmetry. In other disordered models where this degeneracy is not imposed by a symmetry of the model, this degeneracy can only occur accidentally with some probability. The question then is whether this probability of accidental degeneracy between the two preferred exit channels remains finite or becomes rare (i.e. decays to zero) at large times. Within the present strong-disorder RG where the flow is towards some infinite-disorder fixed point, the dominance of the preferred exit channel precisely means that the probability of these accidental degeneracy decays to zero, so that the quasi-equilibrium approximation becomes asymptotically exact at large times. (As discussed in detail in [54] for the case of the Sinai model and in [23] for our present RG procedure in configuration space, the rare events where the quasi-equilibrium approximation breaks down occur with a vanishing probability of order $1/\Gamma = 1/(\ln t)$ at large times.)

4.3. RG rules for the intra-valley energies and entropies

The property of quasi-equilibrium inside each valley discussed above allows us to write RG rules for the intra-valley energies and entropies as we now explain. Within the simplified RG rules where the preferred exit channel $C_{i_{\mathrm{pref}}}$ out of the decimated configuration C^* actually dominates, the decimation of C^* can be interpreted as the merging of the two valleys $C_{i_{\mathrm{pref}}}^{\mathrm{old}}$ and C^* into a single quasi-equilibrated valley $C_{i_{\mathrm{pref}}}^{\mathrm{new}} = C_{i_{\mathrm{pref}}}^{\mathrm{old}} \cup C^*$. The ratio of the times spent in the two sub-valleys C^* and $C_{i_{\mathrm{pref}}}^{\mathrm{old}}$ once they are at equilibrium with each other is given by the ratio of the renormalized transition rates:

$$\lambda \equiv \frac{p_{\mathrm{temporal}}(C^*)}{p_{\mathrm{temporal}}(C_{\mathrm{ipref}}^{\mathrm{old}})} \simeq \frac{W(C_{\mathrm{old}}^{\mathrm{ipref}} \to C^*)}{W(C^* \to C_{\mathrm{old}}^{\mathrm{ipref}})}. \tag{45}$$

Since we have defined the free-energies differences between valleys from this ratio of the renormalized transition rates (equation (12)), one has

$$\lambda = \mathrm{e}^{-\beta(F(C^*) - F(C_{\mathrm{ipref}}^{\mathrm{old}}))}, \tag{46}$$

i.e. the relative weights of the two sub-valleys are related to the free-energy difference of the two sub-valleys.

The full RG rule of equation (35) for the energies of the renormalized valleys in contact with C^* reduces, within the simplified RG rules, to the following renormalization for the energy of the valley $C_{i_{\mathrm{pref}}}$:

$$E(C_{i_{\mathrm{pref}}}^{\mathrm{new}}) = \frac{E(C_{i_{\mathrm{pref}}}^{\mathrm{old}}) + \lambda E(C^*)}{1 + \lambda}. \tag{47}$$

Again, the physical meaning of this rule is very clear: the two sub-valleys are now at equilibrium with each other inside the bigger valley.

Within this quasi-equilibrium picture, it becomes possible to define an intra-valley entropy $S^{\mathrm{intra}}(C)$, and to write the corresponding RG rule when the valley C^* is decimated. With the normalization $p_{\mathrm{temp}}(C^*) + p_{\mathrm{temp}}(C_{\mathrm{old}}^{\mathrm{ipref}}) = 1$ and the ratio of equation (45), the weights of the two sub-valleys are respectively $p_{\mathrm{temp}}(C^*) = \lambda/(1 + \lambda)$ and $p_{\mathrm{temp}}(C_{\mathrm{old}}^{\mathrm{ipref}}) = 1/(1 + \lambda)$. Now taking into account that each sub-valley has its own internal entropy, one obtains that the

internal entropy S^{intra} (defined as $S^{\text{intra}} = -\sum p(c) \ln p(c)$ in terms of the weights $p(c)$ of microscopic configurations c that belongs to the valley) evolves according to

$$S^{\text{intra}}\left(C_{i_{\text{pref}}}^{\text{new}}\right) = \frac{S^{\text{intra}}\left(C_{i_{\text{pref}}}^{\text{old}}\right) + \lambda S^{\text{intra}}(C^*)}{1 + \lambda} + \left[\ln(1 + \lambda) - \frac{\lambda}{1 + \lambda}\ln \lambda\right]. \quad (48)$$

The first term comes from the internal entropies of the sub-valleys, whereas the second term between [...] corresponds to the merging entropy of the two sub-valleys $S_{\text{merging}} = -q \ln q - (1 - q) \ln(1 - q)$ with $q = 1/(1 + \lambda)$.

The valley free-energy that can be defined only from the inter-valley dynamics within the full RG rules can also be expressed in terms of the 'interior' of the valley within the simplified RG rules, because now upon decimation, the decimated valley C^* is attributed to a single surviving valley C_{ipref}. The RG rule for the free-energy simply corresponds to the sum of the partition functions of the two sub-valleys

$$\text{e}^{-\beta F(C_{\text{ipref}}^{\text{new}})} = \text{e}^{-\beta F(C_{\text{ipref}}^{\text{old}})} + \text{e}^{-\beta F(C^*)} = \text{e}^{-\beta F(C_{\text{ipref}}^{\text{old}})}[1 + \text{e}^{-\beta(F(C^*) - F(C_{\text{ipref}}^{\text{old}}))}], \quad (49)$$

i.e. using the definition of λ (equation (46)), one obtains

$$F\left(C_{i_{\text{pref}}}^{\text{new}}\right) = F\left(C_{i_{\text{pref}}}^{\text{old}}\right) - T \ln(1 + \lambda). \quad (50)$$

As it should for consistency, the RG equations for the energy, the entropy and the free-energy are compatible with the thermodynamic relation $F = E - T S^{\text{intra}}$ inside each valley.

4.4. Numerical results for the directed polymer in a two-dimensional medium

The numerical details to follow the 'simplified RG rules' for the directed polymer in a two-dimensional medium are described in our previous works on the non-equilibrium dynamics [22, 23], where we have checked (i) the convergence towards an infinite-disorder fixed point for the distribution of the renormalized barriers, (ii) the validity of the simplified RG rules by comparing the rescaled distributions obtained via the full RG rules and via the simplified RG rules, and where we have measured the barrier exponent ψ from the distribution over samples of the equilibrium time. Here we present our results for the observables characterizing the valleys.
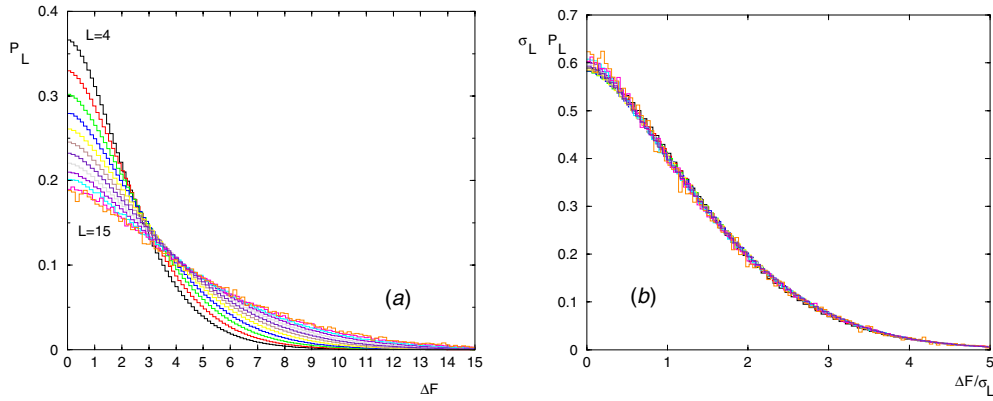
4.4.1. Statistics of the free-energy difference between the two last valleys.
We show in figure 4 the probability distribution $P_L(\Delta F)$ of the free-energy difference $\Delta F = F_2 - F_1$ between the two last valleys of a disordered sample obtained via the simplified RG rules. The rescaled distribution shown in figure 4(b) coincides with the one obtained with the full RG rules (see figure 1(b)). This shows that the simplified RG rules capture correctly the fixed point properties of the valleys.

4.4.2. Statistics of the entropy difference and energy difference between the two last valleys.
We show in figure 5(a) the probability distribution $P_L(\Delta S)$ of the entropy difference $\Delta S = S_2^{\text{intra}} - S_1^{\text{intra}}$ between the two last valleys of a disordered sample: these distributions converge towards a Gaussian shape, in agreement with the droplet scaling theory where the entropy is dominated by a sum of independent small-scale contributions [19]. One then expects that the entropy difference has the central-limit scaling

$$\Delta S_L \simeq L^{1/2} u \quad (51)$$

where u is a Gaussian variable. With our data, we observe the Gaussian distribution, but the sizes $4 \leqslant L \leqslant 18$ are too small to measure precisely the exponent $1/2$.

We show in figure 5(b) the probability distribution $P_L(\Delta E)$ of the energy difference $\Delta E = E_2 - E_1$ between the two last valleys of a disordered sample: as a consequence of

Figure 4. Statistics over the samples of the free-energy difference $\Delta F = F_2 - F_1$ between the two last valleys of a disordered sample using the simplified RG rules. (*a*) Probability distribution $P_L(\Delta F)$ for the sizes $4 \leqslant L \leqslant 15$ (corresponding to $2^L \leqslant 32768$ initial configurations) with a statistics of $n_s \geqslant 5.10^4$ disordered samples (we have data up to $L = 18$ with $n_s = 800$ samples, but histograms are too noisy). (*b*) Same data after the rescaling by the width σ_L : $\sigma_L P_L$ as a function of $\Delta F/\sigma_L$.
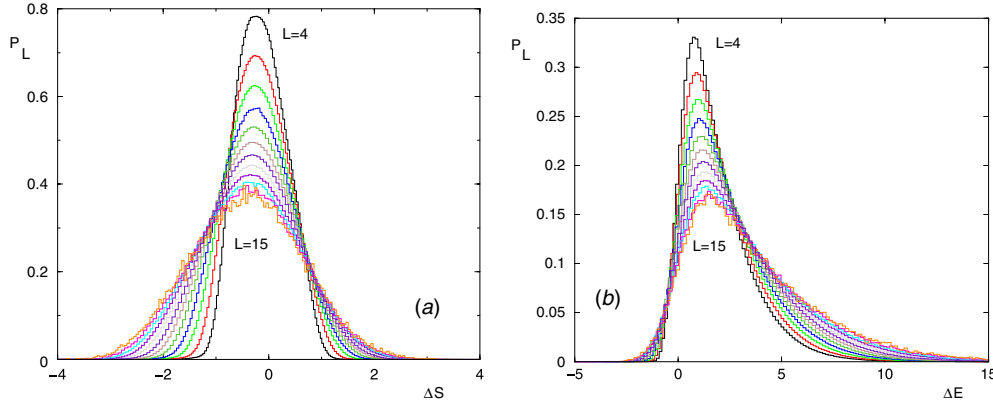
Figure 5. Statistics over the samples of the entropy difference $\Delta S = S_2^{\text{intra}} - S_1^{\text{intra}}$ and of the energy difference $\Delta E = E_2 - E_1$ between the two last valleys of a disordered sample. (*a*) Probability distribution $P_L(\Delta S)$ for the sizes $4 \leqslant L \leqslant 15$: these distribution converge towards a Gaussian shape. (*b*) Probability distribution $P_L(\Delta E)$ for the sizes $4 \leqslant L \leqslant 15$.

the thermodynamic relation $E = F + TS$ inside each valley, the distribution of the energy difference is a convolution of the free-energy distribution of figure 4(*a*) with the Gaussian entropy distribution of figure 5(*a*). Our conclusion is thus that the distributions of the free-energy difference (equation (36)) and of the entropy difference (equation (51)) are the two primary distributions that have good scaling properties even on the small sizes considered here, whereas the probability distribution of the energy is a mixture of these two.

5. Summary and conclusions

From the point of view of 'broken ergodicity' [20, 21] that we have adopted in this paper to describe the equilibrium properties of disordered systems, the appropriate valleys are defined

as the 'components' of configuration space that are separated by large barriers, i.e. the valleys are defined with respect to their capacity to 'confine' the dynamics [20, 21]. In the present paper we have explained how the strong-disorder renormalization procedure allows us to construct in each sample all the valleys that are separated by barriers greater than a prescribed threshold Γ representing the RG scale. To make the link with previous approaches, we have discussed in detail the physical interpretations of the 'full RG' rules and of the 'simplified RG' rules in terms of the 'adiabatic approximation' [49] and in terms of the 'quasi-equilibrium' inside each valley [20], respectively. We have also explained how this explicit RG formulation gives new insights into the main ingredients of the droplet scaling picture. In particular, the 'zero-temperature' nature of the fixed point for the probability distribution of the free-energy difference between valleys, emerges here from a special conservation law along the RG flow for ratios of renormalized transition rates. As an example of application, we have followed numerically the strong-disorder RG rules for the directed polymer in a two-dimensional random medium to obtain the statistical properties of the free-energy difference, the entropy difference and the energy difference between the two longest-lived valleys. In particular, we have obtained that the distribution of the entropy difference converges towards a Gaussian in agreement with the droplet scaling theory where the entropy is dominated by independent small-scales contributions. Our conclusion is that an excitation has two independent primary properties which are (i) its free-energy $\Delta F \sim L^\theta v$ that involves the droplet exponent ($\theta = 1/3$ for the directed polymer in a two-dimensional random medium) and (ii) its entropy $\Delta S \sim L^{d_s/2} u$ that involves the dimensionality d_s of the surface of an excitation ($d_s = 1$ for the directed polymer). These two quantities present nice scaling behaviours even on moderate system sizes, whereas the energy $\Delta E = \Delta F + T \Delta S$ is a mixture of these two scaling behaviours.

A natural question is, of course, whether the renormalization procedure in configuration space that we have developed can be somehow 'projected' in real space. This question is important both numerically and theoretically. From a numerical point of view, it is clear that the formulation of RG rules in configuration space has an exponential 'price', since the number of initial configurations $\mathcal{N}_0$ grows exponentially with the number of degrees of freedom, i.e. grows exponentially with the volume L^d for a system of linear size L in dimension d. This computational complexity is not surprising, since the determination of barriers for the dynamics is expected to be an NP-complete problem [55]. But it is clear that the numerical study of disordered systems via strong-disorder RG rules is then limited to rather small sizes. From a theoretical point of view, as recalled in the introduction, it is usual to describe the equilibrium of statistical physics models by a coarse-graining of some Hamiltonian in real space. As a consequence of locality of the dynamics, the hierarchy of valleys that we construct in configuration space is expected to correspond to some hierarchical organization of real-space structures. It would be very interesting to formulate a consistent RG procedure directly for these real-space structures. But for the moment it is not clear to us what principle should be used to construct these correlated real-space structures recursively in a consistent way.

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